Some Implications of the Density Matrix Deformation in Statistical Mechanics of the Early Universe

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Abstract

This work is an extension of the study into statistical mechanics of the early Universe that has been the subject in prior works of the author, the principal approach being the density matrix deformation. In the work it is demonstrated that the previously derived exponential ansatz may be successfully applied to the derivation of the free and average energy deformation as well as entropy deformation. Based on the exponential ansatz, the derivation of a statistical-mechanical Liouville equation as a deformation of the quantum-mechanical counterpart is presented. It is shown that deformed Liouville equation will possess nontrivial components as compared to the normal equation in two cases: for the original singularity (i.e. early Universe) and for black hole, that is in complete agreement with the results obtained by the author with coworkers in earlier works devoted to the deformation in quantum mechanics at Planck scale. In conclusion some possible applications of the proposed methods are given, specifically for investigation into thermodynamics of black holes.

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1 Introduction

As is known, both statistical mechanics and quantum mechanics of the early Universe (Plank scale)are differing from the well-known analogs [1],[2] due to deformation. The deformation is understood as a theory extension owing to the introduction of one or several additional parameters in such a way that the initial theory appears in the limiting transition [20]. The deformation in quantum mechanics at Planck scale takes different paths: commutator deformation [3], [4],[5],[6] or density matrix deformation [7], [8]. In [9],[10] it has been demonstrated that statistical mechanics of the early Universe, i.e. at Planck scale, may be constructed with the use of the above-mentioned density matrix deformation, but now the density matrix in statistical mechanics. In the present work we proceed with a study of deformation in statistical mechanics of the early Universe. In section 2 the attention is directed to the main features of the Gibbs distribution deformation: using maximum temperature on the order of the Plancks, that is following from the Generalized Uncertainty Relations (GUR) in quantum mechanics, we construct deformation parameter τ; then this parameter is used to introduce the definition of the density pro-matrix into statistical mechanics and to derive the primary implications, including the exponential ansatz giving the statistical density pro-matrix in the explicit form. In section 3 the exponential ansatz is used for derivation of the average energy, free energy and entropy deformations. This ansatz is used further to study deformation of Liouville equation in statistical mechanics. It is demonstrated that normal Liouville equation acquires an additional term in two cases: for the original singularity and for black hole. In conclusion consideration is given to particular applications of the obtained results, specifically in a study into thermodynamics of black holes.

2 Deformed Density Matrix in Statistical Mechanics of the Early Universe

In this section we recall the main features for deformation of the density matrix of the early Universe [9],[10]. To begin, we consider the Generalized Uncertainty Relations "coordinate - momentum" [4],[5],[6]:

$$\triangle x \geq \frac{\hbar}{\triangle p} + \alpha' L_p^2 \frac{\triangle p}{\hbar}. \tag{1}$$

Using relations (1), it is easy to obtain a similar relation for the "energy - time" pair. Indeed (1) gives

$$\frac{\Delta x}{c} \geq \frac{\hbar}{\Delta pc} + \alpha' L_p^2 \frac{\Delta p}{c\hbar}, \tag{2}$$

then

$$\Delta t \geq \frac{\hbar}{\Delta E} + \alpha' \frac{L_p^2}{c^2} \frac{\Delta pc}{\hbar} = \frac{\hbar}{\Delta E} + \alpha' t_p^2 \frac{\Delta E}{\hbar}. \tag{3}$$

where the smallness of L_p is taken into account so that the difference between ΔE and $\Delta(pc)$ can be neglected and t_p is the Planck time $t_p = L_p/c = \sqrt{G\hbar/c^5} \simeq 0,54 \ 10^{-43} sec$. From whence it follows that we have a maximum energy of the order of Planck's:

$$E_{max} \sim E_p$$

Proceeding to the Statistical Mechanics, we further assume that an internal energy of any ensemble U could not be in excess of E_{max} and hence temperature T could not be in excess of $T_{max} = E_{max}/k_B \sim T_p$. Let us consider density matrix in Statistical Mechanics (see [11], Section 2, Paragraph 3):

$$\rho_{stat} = \sum_n \omega_n |\varphi_n >< \varphi_n|, \tag{4}$$

where the probabilities are given by

$$\omega_n = \frac{1}{Q} \exp(-\beta E_n)$$

and

$$Q = \sum_n \exp(-\beta E_n)$$

Then for a canonical Gibbs ensemble the value

$$\overline{\Delta(1/T)^2} = Sp[\rho_{stat}(\frac{1}{T})^2] - Sp^2[\rho_{stat}(\frac{1}{T})], \tag{5}$$

is always equal to zero, and this follows from the fact that $Sp[\rho_{stat}] = 1$. However, for very high temperatures $T \gg 0$ we have $\Delta(1/T)^2 \approx 1/T^2 \geq$

$1/T_{max}^2$. Thus, for $T \gg 0$ a statistical density matrix ρ_{stat} should be deformed so that in the general case

$$Sp[\rho_{stat}(\frac{1}{T})^2] - Sp^2[\rho_{stat}(\frac{1}{T})] \approx \frac{1}{T_{max}^2}, \qquad (6)$$

or

$$Sp[\rho_{stat}] - Sp^2[\rho_{stat}] \approx \frac{T^2}{T_{max}^2}, \qquad (7)$$

In this way ρ_{stat} at very high $T \gg 0$ becomes dependent on the parameter $\tau = T^2/T_{max}^2$, i.e. in the most general case

$$\rho_{stat} = \rho_{stat}(\tau)$$

and

$$Sp[\rho_{stat}(\tau)] < 1$$

and for $\tau \ll 1$ we have $\rho_{stat}(\tau) \approx \rho_{stat}$ (formula (4)) .
This situation is identical to the case associated with the deformation parameter $\alpha = l_{min}^2/x^2$ of QMFL given in section 2 [8]. That is the condition $Sp[\rho_{stat}(\tau)] < 1$ has an apparent physical meaning when:

1. At temperatures close to T_{max} some portion of information about the ensemble is inaccessible in accordance with the probability that is less than unity, i.e. incomplete probability.

2. And vice versa, the longer is the distance from T_{max} (i.e. when approximating the usual temperatures), the greater is the bulk of information and the closer is the complete probability to unity.

Therefore similar to the introduction of the deformed
 quantum-mechanics density matrix in section 3 [8],we give the following
Definition. (Deformation of Statistical Mechanics)
Deformation of Gibbs distribution valid for temperatures on the order of the Planck's T_p is described by deformation of a statistical density matrix (statistical density pro-matrix) of the form

$$\rho_{\mathbf{stat}}(\tau) = \sum_{\mathbf{n}} \omega_{\mathbf{n}}(\tau)|\varphi_{\mathbf{n}} >< \varphi_{\mathbf{n}}|$$

having the deformation parameter $\tau = T^2/T_{max}^2$, where

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1. $0 < \tau \leq 1/4$;

2. The vectors $|\varphi_n >$ form a full orthonormal system;

3. $\omega_n(\tau) \geq 0$ and for all n at $\tau \ll 1$ we obtain $\omega_n(\tau) \approx \omega_n = \frac{1}{Q}\exp(-\beta E_n)$
   In particular, $\lim\limits_{T_{max} \to \infty (\tau \to 0)} \omega_n(\tau) = \omega_n$

4. $Sp[\rho_{stat}] = \sum_n \omega_n(\tau) < 1,\ \sum_n \omega_n = 1$;

5. For every operator B and any τ there is a mean operator B depending on τ

$$< B >_\tau = \sum_n \omega_n(\tau) < n|B|n > .$$

Finally, in order that our Definition agree with the formula (7), the following condition must be fulfilled:

$$Sp[\rho_{stat}(\tau)] - Sp^2[\rho_{stat}(\tau)] \approx \tau. \tag{8}$$

Hence we can find the value for $Sp[\rho_{stat}(\tau)]$ satisfying the condition of Definition 2 (similar to Definition 1):

$$Sp[\rho_{stat}(\tau)] \approx \frac{1}{2} + \sqrt{\frac{1}{4} - \tau}. \tag{9}$$

It should be noted:

1. The condition $\tau \ll 1$ means that $T \ll T_{max}$ either $T_{max} = \infty$ or both in accordance with a normal Statistical Mechanics and canonical Gibbs distribution (4)

2. Similar to QMFL in [7],[8], where the deformation parameter α should assume the value $0 < \alpha \leq 1/4$. As seen from (9), here $Sp[\rho_{stat}(\tau)]$ is well defined only for $0 < \tau \leq 1/4$. This means that the feature occurring in QMFL at the point of the fundamental length $x = l_{min}$ in the case under consideration is associated with the fact that **highest measurable temperature of the ensemble is always $T \leq \frac{1}{2}T_{max}$.**

3. The constructed deformation contains all four fundamental constants: $G, \hbar, c, k_B$ as $T_{max} = \varsigma T_p$, where ς is the denumerable function of α' (1) and T_p, in its turn, contains all the above-mentioned constants.

4. Again similar to QMFL, as a possible solution for (8) we have an exponential ansatz

$$\rho_{stat}^*(\tau) = \sum_n \omega_n(\tau)|n><n| = \sum_n exp(-\tau)\omega_n|n><n|$$

$$Sp[\rho_{stat}^*(\tau)] - Sp^2[\rho_{stat}^*(\tau)] = \tau + O(\tau^2). \tag{10}$$

In such a way with the use of an exponential ansatz (10) the deformation of a canonical Gibbs distribution at Planck scale (up to factor $1/Q$) takes an elegant and completed form:

$$\mathbf{\omega_n(\tau) = exp(-\tau)\omega_n = exp(-\frac{T^2}{T_{max}^2} - \beta E_n)} \tag{11}$$

where $T_{max} = \varsigma T_p$

3 Some Implications

Using in this section only the exponential ansatz of (10), in the coordinate representation we have the following:

$$\rho(x, x', \tau) = \sum_i \frac{1}{Q} e^{-\beta E_i - \tau} \varphi_i(x)\varphi_i^*(x') \tag{12}$$

However, as $H \mid \varphi_i >= E_i \mid \varphi_i >$, then

$$\rho(\beta, \tau) = \frac{1}{Q} \sum_i e^{-\beta H - \tau} \mid \varphi_i >< \varphi_i \mid = \frac{e^{-\beta H - \tau}}{Q}, \tag{13}$$

where $Q = \sum_i e^{-\beta E_i} = Spe^{-\beta H}$. Consequently,

$$\rho(\beta, \tau) = \frac{e^{-\beta H - \tau}}{Spe^{-\beta H}} \tag{14}$$

In this way the deformed average energy of a system is obtained as

$$U_\tau = Sp\rho(\tau)H = \frac{He^{-\beta H - \tau}}{Spe^{-\beta H}} \tag{15}$$

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The calculation of deformed entropy is also a simple task. Indeed, in the general case of the canonical Gibbs distribution the probabilities are equal

$$P_n = \frac{1}{Q} e^{-\beta E_n} \qquad (16)$$

Nevertheless, in case under consideration they are replenished by $exp(-\tau)$ factor and hence are equal to

$$P_n^\tau = \frac{1}{Q} e^{-(\tau + \beta E_n)} \qquad (17)$$

Thus, a new formula for entropy in this case is as follows:

$$S_\tau = -k_B e^{-\tau} \sum_n P_n (ln P_n - \tau) \qquad (18)$$

It is obvious that $\lim_{\tau \to 0} S_\tau = S$, where S - entropy of the canonical ensemble, that is a complete analog of its counterpart in quantum mechanics at Planck scale $\lim_{\alpha \to 0} S_\alpha = S$, where S - statistical entropy in quantum mechanics, and deformation parameter τ is changed by α [8],[10].

Given the average energy deformation in a system U_τ and knowing the entropy deformation, one is enabled to calculate the deformed free energy F_τ as well:

$$F_\tau = U_\tau - T S_\tau \qquad (19)$$

Consider the counterpart of Liouville equation [11] for the unnormed $\rho(\beta, \tau)$ (14):

$$-\frac{\partial \rho(\beta, \tau)}{\partial \beta} = -\frac{\partial}{\partial \beta} e^{-\tau - \beta H} \qquad (20)$$

Since

$$\tau = \frac{T^2}{T_{max}^2} = \frac{\beta_{max}^2}{\beta^2},$$

where $\beta_{max} = 1/k_B T_{max} \sim 1/k_B T_P \equiv \beta_P$, $\tau = \tau(\beta)$. Taking this into consideration and expanding the right-hand side of equation

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(20), we get deformation of Liouville equation further referred to as τ-deformation:

$$-\frac{\partial\rho(\beta,\tau)}{\partial\beta} = -e^{-\tau}\frac{\partial\tau}{\partial\beta} + e^{-\tau}H\rho(\beta) = e^{-\tau}[H\rho(\beta) - \frac{\partial\tau}{\partial\beta}], \qquad (21)$$

where $\rho(\beta) = \rho(\beta, \tau = 0)$.

The first term in brackets (21) generates Liouville equation. Actually, taking the limit of the left and right sides (21) for $\tau \to 0$, we derive the normal Liouville equation for $\rho(\beta)$ in statistical mechanics [11]:

$$-\frac{\partial\rho(\beta)}{\partial\beta} = H\rho(\beta) \qquad (22)$$

By this means we obtain a complete analog of the quantum-mechanical results for the associated deformation of Liouville equation derived in [7],[8],[12]. Namely:

(1)early Universe (scales approximating those of the Plancks, original singularity, $\tau > 0$). The density pro-matrix $\rho(\beta,\tau)$ is introduced and a τ-"deformed" Liouville equation(21), respectively;

(2)after the inflation extension (well-known scales, $\tau \approx 0$) the normal density matrix $\rho(\beta)$ appears in the limit $\lim_{\tau\to 0}\rho(\beta,\tau) = \rho(\beta)$. τ-"deformation" of Liouville equation (21) is changed by a well-known Liouville equation(22);

(3)and finally the case of the matter absorbed by a black hole and its tendency to the singularity. Close to the black hole singularity both quantum and statistical mechanics are subjected to deformation as they do in case of the original singularity [7]–[9],[12],[13]. Introduction of temperature on the order of the Plancks [14],[15] and hence the deformation parameter $\tau > 0$ may be taken as an indirect evidence for the fact. Because of this, the case is associated with the reverse transition from the well-known density matrix in statistical mechanics $\rho(\beta)$ to its τ-"deformation" $\rho(\beta,\tau)$ and from Liouville equation(22) to its τ-"deformation" (21.

4   Conclusion

This paper is an extension of the study presented in [9],[10]. There is a reason to believe that the approaches proposed by the author and

notions of the average energy deformation, free energy deformation, entropy deformation and so on may be applied during investigation into thermodynamics of black holes too [16]. The problem of their relation to the methods used in studies of black holes with due regard for GUR [17]–[19] is of particular importance. Of interest is also elucidation of the following facts: are there any physically meaningful deformations following from the solutions for(8)which differ from the exponential ansatz, or is it unique in a sense? All these problems require further investigation.

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